Consider a random sample \( \left( X_1, X_2, \ldots, X_n \right) \) from a \( N(\mu, 64) \).

(a) Find a best critical region for testing \( H_0 : \mu = 44 \) against \( H_a : \mu = 40 \).

(b) Find \( n \) and \( c \) such that \( \alpha = 0.05 \) and \( \beta = 0.10 \).
Consider a random sample \( (X_1, X_2, \ldots, X_n) \) from a \( \text{Bernoulli}(p) \).

(a) To test \( H_0 : p = 0.5 \) against \( H_a : p = 0.7 \), what is the critical region specified by the likelihood ratio test criterion?

(b) Is this test uniformly most powerful? Explain carefully.

(c) Can \( H_0 \) be rejected at 0.0592 level of significance if a random sample of size 15 yielded \( \sum_{i=1}^{15} X_i = 11 \)? Note that \( \sum_{i=1}^{15} X_i \sim \text{Binomial}(15, p) \)

(d) What is the p-value of this test?

You may want to use Binomial tables for parts (c) and (d).
Let $X \sim N(\mu, 100)$. To test $H_0 : \mu = 80$ against $H_a : \mu < 80$, let the critical region be defined by $C = \{(x_1, x_2, \ldots, x_{25}) : \bar{x} \leq 77\}$, where $\bar{x}$ is the sample mean of a random sample of size 25 from this distribution.

(a) What is the power function $K(\mu)$ of this test?
(b) What is the significant level of this test?
(c) What are the values of $K(80)$, $K(77)$, and $K(74)$?
(d) What is the p-value corresponding to $\bar{x} = 76.52$?
Consider a random sample \( \left( X_1, X_2, \ldots, X_5 \right) \) from a \( \text{Poi}(\lambda), \lambda > 0 \). Suppose we are interested in a Bayes estimator of \( \lambda \) assuming the squared error loss function.

(a) Find the posterior distribution of \( \lambda \) given the data, that is \( p(\lambda | x_1, x_2, \ldots, x_5) \) if the prior distribution of \( \lambda \) is \( h(\lambda) = 4\lambda^2 e^{-2\lambda} \), \( \lambda > 0 \). Assume \( \sum_{1}^{5} X_i = 4 \).

(b) Find the Bayes estimator of \( \lambda \) with respect to the squared error loss function.

(c) Show that the Bayes estimator is a weighted average of the MLE of \( \lambda \) and the prior mean with weights \( \frac{5}{7} \) and \( \frac{2}{7} \) respectively.
5  (a) Let $X$ have a normal distribution with parameter $\theta$, $\left( \theta = \frac{1}{\sigma^2} \right)$. That is

$$f(x) = \frac{\sqrt{\theta}}{\sqrt{2\pi}} e^{-\frac{\theta x^2}{2}}.$$  Let the prior of $\theta$ be Exp($\beta$). That is

$$p(\theta) = \frac{1}{\beta} e^{-\frac{\theta}{\beta}}, \beta > 0.$$  Find the posterior distribution and its mean.

(b) Find the predictive distribution of $X$. 