Mathematical Statistics
Spring 2005
Test 4 (Closed Book)
Name:…………………………………………………………

(5+5+6)+(8+8)+(6+4+4+2)+(8+8+4)+(6+4+3+3)+(8+4+4) = 100

1 Consider a random sample $(X_1, X_2, ..., X_n)$ from a Geometric distribution with parameter p.

(a) Show that $S = \sum_{i=1}^{n} X_i$ is a sufficient statistic for p.

(b) Show that the distribution of $S = \sum_{i=1}^{n} X_i$ is Negative Binomial.
(Hint: Use m.g.f.)

(c) Show that the conditional distribution of $(X_1, X_2, ..., X_n)$ given S is independent of p.

2 Consider a random sample $\left(X_1, X_2, \ldots, X_n\right)$ from a $N(\mu, 64)$.

(a) Find a best critical region for testing $H_o : \mu = 44$ against $H_a : \mu = 40$.

(b) Find n and c such that $\alpha = 0.05$ and $\beta = 0.10$.

3 Consider a random sample $(X_1, X_2, ..., X_n)$ from a $Bernoulli(p)$.

(a) To test $H_o : p = 0.5$ against $H_a : p = 0.7$, what is the critical region specified by the likelihood ratio test criterion?

(b) Is this test uniformly most powerful? Explain carefully.

(c) Can H_o be rejected at 0.0592 level of significance if a random sample of size 15 yielded $\sum_{i=1}^{15} X_i = 11$? Note that $\sum_{i=1}^{15} X_i \sim Binomial(15, p)$

(d) What is the p-value of this test?

You may want to use Binomial tables for parts (c) and (d).

4 Consider a random sample $(X_1, X_2, ..., X_n)$ from a $b(1, p)$ and let $\tau(p) = p$.

(a) Find the Cremer-Rao lower bound for the variance of any unbiased estimator of $\tau(p)$.

(b) Is $\overline{X}$ UMVUE of $\tau(p)$? Explain.

(c) Find the efficiency of the estimate $\overline{X}$.

5 Let $X \sim N(\mu,100)$. To test $H_o : \mu = 80$ against $H_a : \mu < 80$, let the critical region be defined by $C = \{(x_1, x_2, ..., x_{25}) : \overline{x} \le 77\}$, where $\overline{x}$ is the sample mean of a random sample of size 25 from this distribution.

(a) What is the power function $K(\mu)$ of this test?

(b) What is the significant level of this test?

(c) What are the values of $K(80)$, $K(77)$, and $K(74)$?

(d) What is the p-value corresponding to $\overline{x}$ = 76.52?

6 Consider a random sample $\left(X_1, X_2, ..., X_5\right)$ from a $Poi(\lambda)$, $\lambda > 0$. Suppose we are interested in a Bayes estimator of λ assuming the squared error loss function.

(a) Find the posterior distribution of λ given the data, that is $p(\lambda \mid x_1, x_2, ..., x_5)$ if the prior distribution of λ is $h(\lambda) = 4\lambda^2 e^{-2\lambda}$, $\lambda > 0$. Assume $\sum_1^5 X_i = 4$.

(b) Find the Bayes estimator of λ with respect to the squared error loss function.

(c) Show that the Bayes estimator is a weighted average of the MLE of λ and the prior mean with weights $\frac{5}{7}$ and $\frac{2}{7}$ respectively.