Probability and Statistics
Test 4

Name:………………………………………..

16+10+4+12+12+12+10+12+12=100

Read the questions carefully. In question (1), you have to completely characterize the distribution. For example, a normal distribution is complete when you give the mean and the variance with the name of the distribution. For some other distributions you can give the name and the degrees of freedom.

Sample mean $\bar{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$

Sample variance $S^2 = \dfrac{\sum_{i=1}^{n} \left(X_i - \bar{X} \right)^2}{n-1}$

You can use the calculator only for calculations.

1 (a) If $Z \sim N(0,\ 1)$, then what is the distribution of $Y = Z^2$?

(b) If $X \sim N(\mu,\ \sigma^2)$, then what is the distribution of $U = X - \mu$?

(c) If $X \sim N(\mu,\ \sigma^2)$, then what is the distribution of $V = \frac{X}{\sigma}$?

(d) If $X_1,\ X_2, \ldots, X_n$ i.i.d.$\sim N(\mu,\ \sigma^2)$, then what is the distribution of the sample mean $\bar{X}$?

(e) If $X_1,\ X_2, \ldots, X_n$ i.i.d.$\sim N(\mu,\ \sigma^2)$, then what is the distribution of $\frac{(n-1)S^2}{\sigma^2}$?

(f) If $X_1,\ X_2, \ldots, X_n$ i.i.d.$\sim N(\mu,\ \sigma^2)$, then what is the distribution of $\frac{\sum_{i=1}^{n}(X_i - \mu)^2}{\sigma^2}$?

(g) If $X_1 \sim N(\mu_1,\ \sigma_1^2)$, $X_2 \sim N(\mu_2,\ \sigma_2^2)$, and X_1 and X_2 are independent, then what is the distribution of $Y = a_1X_1 + a_2X_2$?

(h) If $X_1,\ X_2, \ldots, X_n$ are a random sample from a distribution with mean μ and variance σ^2, then what statistics will give us an approximately normal distribution?

2 Prove that, if $X \sim N(\mu, \sigma^2)$, then $Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$.

3 Find $Z_{.05}$ and $\chi^2_{0.05,(1)}$. What is the relationship between $Z_{.05}$ and $\chi^2_{0.05,(1)}$

4 If $X \sim N(650, 25^2)$, then find

(a) $P(620 < X < 700)$.

(b) $P(|X-650| > 25.0)$.

(c) a constant c such that $P(|X-650| < c) = 0.8664$.

5 Let $X_1,\ X_2,\ldots,\ X_{16}$ be a random sample from a distribution with moment generating function of $M_X(t)=\exp\left(46.58\,t+20.48\,t^2\right)$.

Find $P\left(710.72\le\sum_{i=1}^{16}X_i\le783.68\right)$.

6 Let X be a random variable with mean 54.03 and standard deviation 5.8. Let $\bar{X}$ be the sample mean of a random sample of size 49 from this distribution. Find $P\left(53\le\bar{X}\le56\right)$.

7 If $Z_1,\ Z_2$ are i.i.d $N(0,\ 1)$, then prove that the distribution of $U=Z_1^2+Z_2^2$ is $\chi^2_{(2)}$.

8 Let $X_1,\ X_2,\ldots,X_{16}$ be a random sample form $N\left(100,8^2\right)$. Find the following:

(a) c such that $P\left(S^2 > c\right) = 0.05$.

(b) $P\left[\sum_{i=1}^{16}\left(X_i - 100\right)^2 > 371.968\right]$

9 Suppose that the distribution of the weight of a prepackaged "1-pound" bag of carrots is $N\left(1.16, 0.07^2\right)$ and the distribution of the weight of a prepackaged "3-pound" bag of carrots is $N\left(3.22, 0.08^2\right)$. Selecting bags at random, find the probability that the sum of three 1-pound bags exceeds the weight of one 3-pound bag.